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TRANSFORMATION OF GEODETIC HEIGHTS BETWEEN LOCAL REFERENCE SYSTEMS – ALGORITHM FOR RIGOROUS ADJUSTMENT

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Summary

The topic of transformations between planar or spatial coordinate systems has been extensively addressed in the literature for years. Usually, researchers present in scientific papers the definitions of iterative algorithms or analytic solutions of 2D or 3D transformations. However, there is a gap in the field with regard to 1D (vertical) transformations. It seems to be quite easy to fill, as it is sufficient to determine one parameter – the vertical shift, i.e. the height difference between two local reference levels. For this purpose, a single (at least) adjustment point is needed, i.e. a surveying benchmark of known heights in both reference systems. However, there is no precisely defined model of rigorous adjustment for a larger number of adjustment points ($s > 1$). In this paper, the Authors' have shown several variants of transformations between vertical coordinate systems. These variants include different approaches to weighting the "observations" (heights of adjustment points), such as transformation without weighting and transformation with weighting dependent on the distance between adjustment points (horizontal and vertical distances). Each of the variants was developed in two successive approaches: without transformation corrections and with post-transformation corrections. The research arrived at the latter analogically to the corrections used in planar coordinate transformations (a modification of post-transformation Hausbrandt correction). The analyses made it possible to draw general conclusions determining the relationships between weighting the observations together with applying post-transformation corrections, and the results of height transformation. These findings can become the basis for developers of geodetic computing systems, in terms of the possibility of extending them with a 1D transformation module (in addition to 2D and 3D transformations).

Keywords

1D transformation • post-transformation corrections • transformation weighting

1. Introduction

The concept of transformation usually means the mathematical operation of converting rectangular coordinates between different systems. We can distinguish between 2- or 3-dimensional (planar or spatial) transformations. Solving a transformation problem involves determining the values of the parameters defining the functional relationships between the initial (primary) coordinate system and the target (secondary) system. In typical geodetic calculations (e.g. for horizontal control networks), the Helmert conformal transformation (preserving a constant angle) is usually used (Teunissen 1988; Watson 2006; Öcalon 2018). For converting the coordinates on a plane (*xy*), we refer to a 4-parameter transformation (two components of vector displacement between systems, an angle of rotation, and a scale change coefficient), while in 3D space (*xyz*) it is necessary to use a 7-parameter transformation (three components of the vector displacement, three angles of rotation, and a scale change coefficient). The solutions to 2D and 3D transformations involve solving nonlinear systems of equations by the least squares method, in which the rotation matrix is orthonormal. The 2D transformation is subject to linear transformations and can be solved using classical methods. The 3D transformation, in addition to linearization, often requires iterative algorithms (Zeng and Yi 2011; Zeng et al. 2016) or analytical algorithms (Shen et al. 2006; Zeng and Yi 2011; Zeng 2015) that result in the loss of the orthonormal matrix characteristic (Sjöberg 2013). For these reasons, both the 2D and 3D transformations have been extensively studied in the literature.

In theoretical studies on coordinate transformations (Jaworski et al. 2022; Mendel 2011; Chen and Hill 2005; Gargula 2004) there is no definition of a 1-dimensional transformation – for converting the heights of points set with respect to two different reference levels. This task seems quite simple, as it suffices to determine a single parameter – the vertical shift, i.e. the difference in height between two local reference levels. For this purpose, a single (at least) adjustment point is needed, that is, a reference with a known (determined) height in both systems. In the case of a larger number of adjustment points, the problem of overdetermination (mathematical contradiction) arises, which, according to the principles of geodetic calculations, requires carrying out a rigorous adjustment of the transformation, i.e. determining corrections for the adjustment points.

This paper proposes an algorithm for rigorous adjustment of a 1-dimensional (height) transformation. The practical part, which includes the numerical verification of the developed algorithm, focuses on different possibilities of weighting the adjusted values, i.e. the heights of the adjustment points.

2. Mathematical notation of the 1-dimensional transformation adjustment model

Mathematical notation of the 1-dimensional transformation adjustment model demonstrates the initial transformation equation (1):

$$H_i^{(W)} = H_i^{(P)} + H_0 \tag{1}$$

where:

i – ordinal designation of an adjustment point ($i = 1, 2, \ldots, s$; s – number of all adjustment points),

(W) – designation of the secondary reference system,
(P) – designation of the primary reference system,
H_0 – vertical shift-vector (the height of the primary system reference level expressed in the secondary system – see Fig. 1)

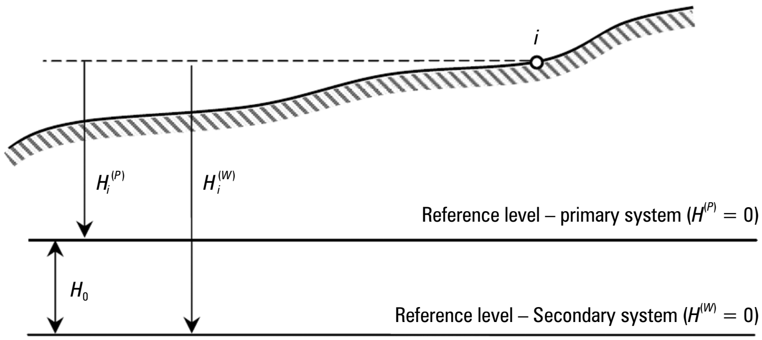

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Fig. 1. Principle of vertical transformation

Given that the number of adjustment points s > 1, the heights of these points (in the secondary system) should undergo adjustment corrections:

$$H_i^{(W)} + v_i = H_i^{(P)} + \left(\tilde{H}_0 + \delta H_0\right) \tag{2}$$

where:
$\tilde{H}_0$ – approx. value of the H_0 shift parameter,
δH_0 – a sought increase of height transformation parameter.

The approximate value of the transformation parameter is calculated on the basis of known heights of adjustment points in both systems, for example:

$$\tilde{H}_0 = \frac{1}{s} \cdot \sum_{i=1}^{s} \left(H_i^{(W)} - H_i^{(P)}\right) \tag{3}$$

Transforming equation (2) we obtain the general equation for the corrections of direct "observation" – with one parameter δH_0:

$$v_i = \delta H_0 + l_i \tag{4}$$

where:
l_i – free term,

$$l_i = H_i^{(P)} + \tilde{H}_0 - H_i^{(W)} \tag{5}$$

Corrections (4) should be calculated under the assumption that adjusted values (1) are non-uniformly exact. The weights should be assumed as values depending on mutual distances between the adjustment points (it is necessary to know the approximate coordinates: x_i, y_i). Two suggestions (at choice) are shown below on how to determine the weight for the i^{th} adjustment point – as a value inversely proportional to:

1) the horizontal distance of the point from the "centre of gravity" of the set of adjustment points.

$$p_i^{(1)} = \frac{1}{\sqrt{\left(x_i - \frac{1}{s} \cdot \sum_{i=1}^{s} x_i\right)^2 + \left(y_i - \frac{1}{s} \cdot \sum_{i=1}^{s} y_i\right)^2}} \tag{6}$$

2) the average horizontal distance between the i point and the restthe of adjustment points.

$$p_i^{(2)} = \frac{1}{\frac{1}{s-1} \cdot \left(d_{i,i+1} + d_{i,i+2} + \ldots + d_{i,s}\right)} \tag{7}$$

The system of adjustment equations (4) will be written in a matrix form:

$$\mathbf{V} = \mathbf{A} \cdot \boldsymbol{\delta}\mathbf{X} - \mathbf{L} \tag{8}$$

where:

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_s \end{bmatrix}_{s,1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{s,1} \cdot \left[\delta H_0\right]_{1,1} - \begin{bmatrix} \left(H_1^{(W)} - H_0 - H_1^{(P)}\right) \\ \left(H_2^{(W)} - H_0 - H_2^{(P)}\right) \\ \vdots \\ \left(H_s^{(W)} - H_0 - H_s^{(P)}\right) \end{bmatrix}_{s,1}$$

The system of equations (8) satisfies the least squares condition (for non-uniformly exact values):

$$\mathbf{V}^{\mathrm{T}} \cdot \mathbf{P} \cdot \mathbf{V} = \mathbf{min.} \tag{9}$$

where:

$$\boldsymbol{\delta}\mathbf{X} = \delta H_0 = (\mathbf{A}^{\mathrm{T}} \cdot \mathbf{P} \cdot \mathbf{A})^{-1} \cdot \mathbf{A}^{\mathrm{T}} \cdot \mathbf{P} \cdot \mathbf{L} \tag{10}$$

After substituting the calculated parameter (10) into equation (8), we obtain the correction values to be added to the heights of the adjustment points in the secondary system:

$$\bar{H}_i^{(W)} = H_i^{(W)} + v_i \tag{11}$$

According to the notation of equations (1) and (2), the final value of the shift parameter of the vertical systems will be:

$$H_0 = \tilde{H}_0 + \delta H_0 \tag{12}$$

Then, based on the parameter (12), we can calculate the heights of points (j) transformed from the primary system to the secondary system:

$$H_j^{(W)} = H_j^{(P)} + H_0 \tag{13}$$

where:

j – the ordinal designation of a transformation point ($j = 1, 2, \ldots, t$; t – the number of all transformation points).

The heights of adjustment points calculated according to (11) should be equal to the heights of points calculated according to equation (13), which will confirm the correctness of the adjustment process carried out:

$$\bar{H}_i^{(W)} = H_i^{(P)} + H_0 \tag{14}$$

Matrix equation (10) is reduced to the calculation of the weighted arithmetic mean:

$$\delta H_0 = \frac{\sum_{i=1}^{s} p_i \cdot \left(H_i^{(P)} + \tilde{H}_0 - H_i^{(W)} \right)}{\sum_{i=1}^{s} p_i} \tag{15}$$

Given the above (15), we will use the equation for the mean error of the arithmetic mean (law of transmission of mean errors – lit.) to assess the accuracy after the adjustment:

$$m_{H_0} = \pm \frac{m_0}{\sqrt{\sum_{i=1}^{s} p_i}} \tag{16}$$

where:

m_0 – unit mean error of determination l_i (5),

$$m_0 = \pm \sqrt{\frac{\mathbf{V}^{\mathrm{T}} \cdot \mathbf{P} \cdot \mathbf{V}}{s-1}} \tag{17}$$

The value calculated according to equation (16) should be regarded as the mean error in the height of the adjustment points after the transformation (in the secondary system).

The equations given in this chapter are mostly the authors' solution, except for the equations using the classical least squares method (8), (9) and the post-adjustment accuracy assessment (16), (17) – see e.g. (Wisniewski 2005).

3. Post-transformation corrections for heights of transformation points

During the transformation of rectangular *xy* coordinates often arises the problem of incompatibility of corrected (adjusted) coordinates of the secondary adjustment points with their catalogue (archive) values. The problem becomes significant when the adjustment points function as points of a geodetic (e.g. detailed) control network, whose coordinates are "well determined" and should not be changed. In such a case, the catalogue coordinates (of the secondary system adjustment points) are left unchanged, while the so-called post-transformation Hausbrandt corrections are applied to the transformed points (Hausbrandt 1971; Beluch 2009; Swiętoń 2012).

A similar problem to the one described above can also occur for the heights of the adjustment points (inconsistency of the corrected heights of the adjustment points with their catalogue values). The solution to the problem may be to apply similar post-transformation corrections as in the case of horizontal coordinates. These corrections will consist of restoring the heights of the adjustment points in the secondary system to their pre-adjustment values:

$$H_i^{(W)} = \bar{H}_i^{(W)} - v_i \tag{18}$$

Next, based on adjustment corrections v_i (for adjustment points), we will calculate the corrections v_j for heights of transformation points:

$$v_j = \frac{\sum_{i=1}^{s}\sum_{j=1}^{t}\left(v_i \cdot \frac{1}{d_{ij}^2}\right)}{\sum_{i=1}^{s}\sum_{j=1}^{t}\left(\frac{1}{d_{ij}^2}\right)} \tag{19}$$

where:

d_{ij} – the distance of the $^{j\text{th}}$ transformation point from the i^{th} adjustment point (it is necessary to know the approximate *xy* coordinates of transformed vertical points).

As a result, the heights of transformed points in the secondary system (13) obtain the final values:

$$\bar{H}_j^{(W)} = H_j^{(W)} + v_j \tag{20}$$

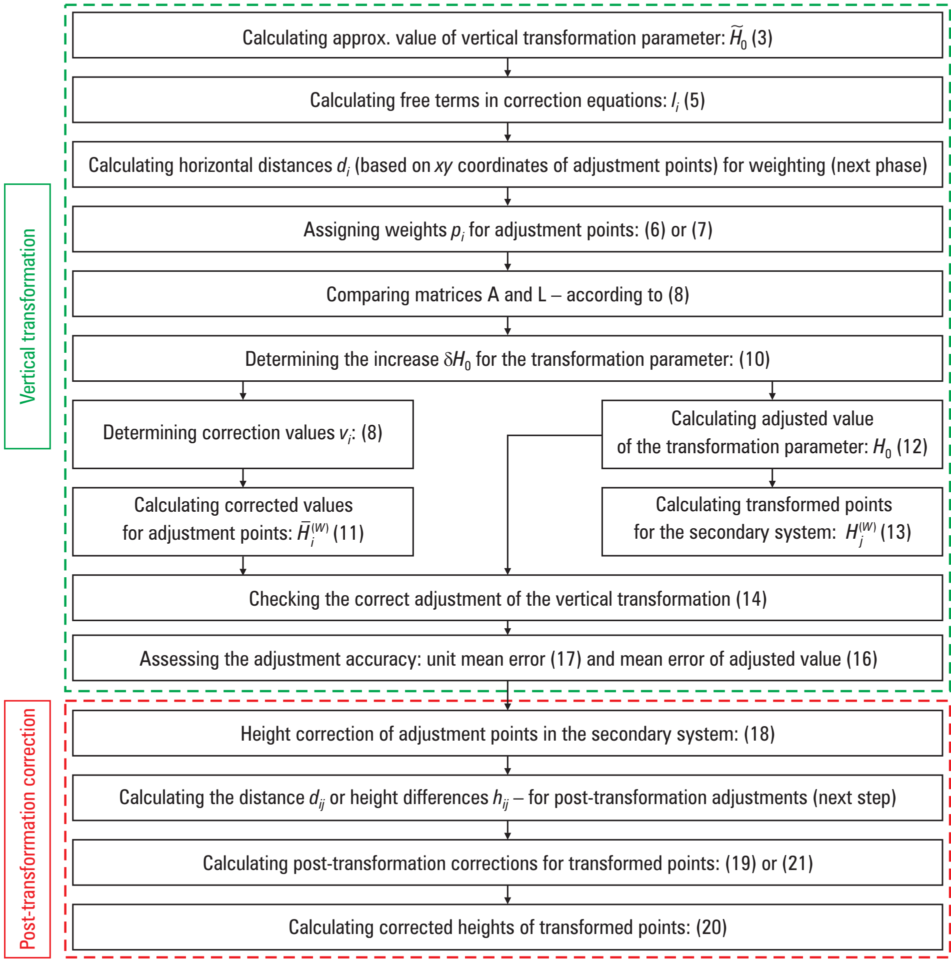
The possibility of introducing height differences h_{ij} between the transformed points (*j*) and the adjustment points (*i*) – instead of distance d_{ij} – into equation (19) remains an open question:

$$v_j^{(2)} = \frac{\sum_{i=1}^{s}\sum_{j=1}^{t}\left(v_i \cdot \frac{1}{h_{ij}^2}\right)}{\sum_{i=1}^{s}\sum_{j=1}^{t}\left(\frac{1}{h_{ij}^2}\right)} \tag{21}$$

The impact of the applied type of post-transformational corrections (e.g. (19) or (21)) on the final results of the transformation will be verified and evaluated numerically using a practical example (in the further part of the paper).

4. Scheme of the computational algorithm

Figure 2 shows a detailed scheme of the computational algorithm for the 1-dimensional transformation for $s > 1$ adjustment points (with reference to the relevant equations). The algorithm proposed by the authors consists of two steps: (1) the vertical transformation with aligning the adjustment points (as non-uniformly accurate values); (2) the post-transformation correction.



Source: Authors' own study

Fig. 2. Scheme of the computational algorithm

5. Numerical example

The adjustment method for height transformations of points that was proposed in this paper was tested on the data (Table 1).

Table 1. Input data for the 1-dimensional transformation between systems

Nr	Horizontal coordinates		Height system:	
	System 2000		primary	secondary
	X [m]	Y [m]	$H^{(P)}$ [m]	$H^{(W)}$ [m]
1*	5537981.38	7431695.46	338.258	290.233
2*	5537932.65	7431738.58	342.190	294.150
3*	5537987.91	7431786.09	334.584	286.561
101	5537920.01	7431796.92	348.020	–
102	5537950.50	7431815.57	343.961	–
103	5537965.13	7431775.19	336.375	–
104	5537983.58	7431742.89	336.140	–
105	5537941.43	7431787.35	341.870	–

* adjustment points

The calculations were carried out in several variants – depending on how the height of the adjustment points was weighted (6), (7) and how post-transformation adjustments were taken into account (19), (21).

In the initial variant (variant 1), the transformations were performed without weighting the observations. In the proposed variant 1, the results of the 1-dimensional transformations featured inconsistencies between the calculated (as a result of the rigorous adjustment of transformations) heights of the adjustment points and their catalogue values (assumed as unambiguously determined and with no corrections). In order to restore the adjustment points to their catalogue values in the secondary system, and at the same time to calculate the remaining heights of the points in the catalogue system, post-transformation corrections were introduced for the adjustment points. For comparison, two variants for calculating the adjustments were used: variant (a), in which the corrections depended on the horizontal distances of the j^{th} transformed point from the i^{th} adjustment point (19), and variant (b), in which the corrections depended on the height difference of the j^{th} transformed point from the i^{th} adjustment point (21). The results of the 1-dimensional transformations in variants 1, 1a and 1b are included in Table 2.

Table 2. Results of the 1-dimensional transformations according to the proposed algorithm – Variant 1

Nr	Variant 1		Variant 1a		Variant 1b	
	$H^{(W)}$ [m]	v_i [m]	$H^{(W)}$ [m]	v_j [m]	$H^{(W)}$ [m]	v_j [m]
1*	290.229	−0.0043	290.233	–	290.233	–
2*	294.161	0.0107	294.150	–	294.150	–
3*	286.555	−0.0063	286.561	–	286.561	–
101	299.991	m_0 [m] = 0.0093	299.993	0.0024	299.993	0.0026
102	295.932		295.930	−0.0021	295.937	0.0054
103	288.346		288.343	−0.0029	288.342	−0.0032
104	288.111	m_{H0} [m] = 0.0054	288.110	−0.0009	288.107	−0.0034
105	293.841		293.842	0.0010	293.850	0.0088

In the second variant (variant 2), it was assumed in the transformations that weights will be equal to the value inversely proportional to the horizontal distance of the point from the "centre of gravity" of the set of adjustment points. As with the transformations in the variant 1, the results of the calculations in the variant 2 required post-transformation adjustments. These were made by following the same procedure as the calculations of the variant 1. The results of the 1-dimensional transformation in variants 2a and 2b are included in Table 3.

Table 3. Results of the 1-dimensional transformation according to the proposed algorithm – Variant 2

Nr	Variant 2		Variant 2a		Variant 2b	
	$H^{(W)}$ [m]	v_i [m]	$H^{(W)}$ [m]	v_j [m]	$H^{(W)}$ [m]	v_j [m]
1*	290.227	−0.0056	290.233	–	290.233	–
2*	294.159	0.0094	294.150	–	294.150	–
3*	286.553	−0.0076	286.561	–	286.561	–
101	299.989	m_0 [m] = 0.0015	299.991	0.0011	299.991	0.0014
102	295.930		295.927	−0.0033	295.935	0.0041
103	288.344		288.340	−0.0041	288.340	−0.0044
104	288.109	m_{H0} [m] = 0.0056	288.107	−0.0022	288.105	−0.0047
105	293.839		293.839	−0.0003	293.847	0.0076

Applying post-transformation corrections in variants 2a and 2b resulted in differences in the heights of the transformed points, with a standard deviation of ±7 mm. The maximum difference between the post-transformation coordinates and coordinates after including the corrections amounted to 11 mm, while the smallest difference – 2 mm.

In the third transformation variant (variant 3), the weights were taken as values inversely proportional to the horizontal distance between point *i* and the other adjustment points. For the results of the 1-dimensional transformations in the proposed variant 3 of weights, two variants for the calculation of post-transformation adjustments were also applied (as for the variants 1 and 2): the variant (a) and the variant (b). The results of the 1-dimensional transformations in variants 3, 3a and 3b are included in (Table 4).

Table 4. Results of the 1-dimensional transformations according to the proposed algorithm – Variant 3

Nr	Variant 3		Variant 3a		Variant 3b	
	$H^{(W)}$ [m]	v_i [m]	$H^{(W)}$ [m]	v_j [m]	$H^{(W)}$ [m]	v_j [m]
1*	290.228	−0.0049	290.233	–	290.233	–
2*	294.160	0.0101	294.150	–	294.150	–
3*	286.554	−0.0069	286.561	–	286.561	–
101	299.990	m_0 [m] = 0.0011	299.992	0.0018	299.992	0.0021
102	295.931		295.929	−0.0026	295.936	0.0049
103	288.345		288.342	−0.0034	288.341	−0.0037
104	288.110	m_{H0} [m] = 0.0055	288.109	−0.0014	288.106	−0.0039
105	293.840		293.841	0.0005	293.848	0.0083

Applying different post-transformation corrections in the variant 3(a) and the variant 3(b) led to differences in the heights of the transformed points – with the same standard deviation as in the variant 2. Similarly, the maximum difference in coordinates after transformation and after corrections was equal to 11 mm, while the smallest difference was 2 mm.

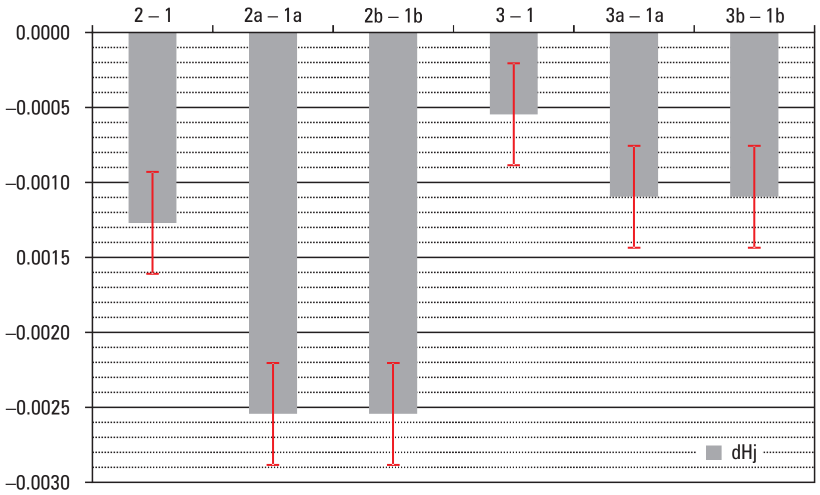
The results of transformations carried out in the variants that weighted the observations (variants 2 and 3, denoted by $H_j^{(2)\text{ or }(3)}$) were compared to the results of transformations without weights (the variant 1, denoted by $H_j^{(1)}$). Height coordinates of the points after the transformation and after applying the same post-transformation corrections in the differentiated pairs were subjected to differentiation analyses:

$$dH_j = H_j^{(2)\text{ or }(3)} - H_j^{(1)} \tag{22}$$

The sets of vertical coordinate differences in the comparison pairs were shared by the same value due to the feature of the 1-dimensional transformation, in which the unknown is a single parameter of displacement of the object as a whole. Vertical coordinate differences in the comparison pairs are included in Table 5 and illustrated in Figure 3.

Table 5. The values of differences of vertical coordinates in the transformation variants with and without weighting

Compared pair	dH_j [m]
2 – 1	–0.0013
2a – 1a	–0.0025
2b – 1b	–0.0025
3 – 1	–0.0005
3a – 1a	–0.0011
3b – 1b	–0.0011



Source: Authors' own study

Fig. 3. Coordinate differences of transformed points between 3 variants of weighting (standard error bars are marked on the figure)

A comparison of the transformation results in the variants with and without weighting highlighted the influence on the transformation result of both the way the weights were adopted and the applied post-transformation adjustments. The application of a specific observation weighting approach ((a) or (b)) alters the transformation results within a range of ±12 mm. The application of post-transformation corrections in transformation variants 2 and 3 affects the differences between the transformed heights, on average doubling their value compared to the values without corrections. However, this procedure is necessary to express the heights of the transformed points in the pattern defined by the pre-transformation adjustment points.

The computational variants presented in this chapter are primarily test cases for the developed equations (Chapter 2). Nonetheless, the selection of an appropriate variant of the height transformation equation should depend on both the number of adjustment points and their mutual distribution. In the case of a larger number of adjustment points and their uneven distribution, the use of weights (variants 2 or 3) is proposed. However, regardless of the weights, post-transformation corrections (sub-variants "a" or "b") should be introduced if the adjustment points have "well" determined (reliable) heights in the secondary system that should not be distorted by the adjustment. Variants marked with the letter "a" are recommended for differentiated distances between adjustment points, while variants "b" are recommended for significant height differences between adjustment points.

6. Summary and conclusions

The algorithm for conducting 1-dimensional transformations with $s > 1$ adjustment points proposed in the paper gives a rigorous solution for calculating the heights of points relative to two different reference levels. In the solution proposed by the authors, adopting the adjusted heights of the adjustment points as non-uniformly exact values for which the weights should depend on the mutual distances between the adjustment points (situational or height). Following the Helmert flat transformation, it is also suggested to use the post-transformation correction, which would not change the heights of the adjustment points in the secondary system (which is also the target system for the transformed points). The developed computational algorithm was tested for several comparative variants. The analysis of the results allows for the following conclusions:

1. The use of a specific approach to observation weighting affects the change in the results of the transformation by the average error of the transformation m_0,
2. The use of post-transformation corrections in transformation variants affects the differences in transformed coordinates by approximately doubling their differences compared to the values without corrections.

The results of research included in this paper can serve as a proposal for creators of geodetic computational systems, in terms of the possibility of extending them with a 1D transformation module (in addition to 2D and 3D transformations).

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