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SIMULATING TRANSIENT FLOW IN UNSATURATED POROUS MEDIA USING FINITE-DIFFERENCE MODELING

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Summary

The purpose of this study concerns the establishment of numerical model of transient flow in a variably saturated porous medium. Groundwater flow can only be studied adequately if one considers the fluxes between the saturated and the unsaturated zones through the free surface. However, this water table undergoes variations in level resulting either from losses of mass by gravity drainage or evaporation or from an excess of mass by infiltration from the surface of the porous medium. This describes the various phenomena that groundwater flow can undergo, such as gravity drainage, infiltration and evaporation. The adopted model is based on the Richards equation, which is a parabolic and strongly non-linear equation. The h-based form of the Richards equation is solved numerically by using the 1D upwind finite difference method. Referring to published experimental work and comparing our numerical results with their results, we have obtained a good fit. The importance of this model lies in its simplicity and its generality in treating the different flow states in a variably saturated porous medium, and therefore its usefulness in practice for a wide range of applications, contributing significantly to the understanding of transient flow phenomena in variably saturated porous media. Its capacity to address the complexities of groundwater movement, including gravity-driven drainage, infiltration, and evaporation, underlines its versatility and its potential to make meaningful contributions to various scientific and engineering fields.

Keywords

transient • partially saturated • hydraulic head • drainage • infiltration

1. Introduction

Groundwater flow can only be studied adequately if one considers the fluxes between the saturated zone and the unsaturated zone through the free surface. However, this water table undergoes variations in level resulting either from losses of mass by gravity drainage or evaporation or from an excess of mass by infiltration from the ground's surface.

In recent decades, the hydraulics of unsaturated soils have received significant attention from scientists in various fields such as geotechnics, hydrogeology, agronomy...

etc. This interest was driven by the desire to find solutions to the problems associated with water flow in partially saturated soils. Liakopoulos [1965] and Baver et al. [1972] have conducted detailed studies on the transfer mechanism in partially saturated soils.

However, the difficulty of acquiring in situ data relating to flow parameters in unsaturated soils, encouraged studying this type of transient flow through numerical modeling, which flourished thanks to advancements in applied mathematics and computer science. Thus, the study of flows in partially saturated media has progressed a lot since the 1970s, a period when models based on numerical simulation began to be designed. In literature, many numerical models have affected the transient water flow mechanism in unsaturated soils. For comparison, Khire et al. [1997] conducted research using two software programs, HELP and UNSATH, both designed for studying water flow in partially saturated porous media. Their findings showed that HELP predicted too much percolation, while UNSATH predicted too little. The relevance of pollutant migration is evident in various aspects of water resource management and environmental sustainability. In recent years, pollutant migration processes have also been studied, as exemplified by the work of Bennacer et al. [2022]. While this type of studies concentrates on pollutant migration in saturated conditions, it is essential to recognize that addressing pollutant transport in unsaturated porous media introduces a distinct set of challenges and considerations. This underlines the need for a comprehensive understanding of pollutant migration in unsaturated environments to address environmental concerns and ensure effective water resource management. Other numerical models have been used for the modeling of multi-phase flows based on the finite difference method or finite element method [Helmig and Huber 1998, Bastian and Helmig 1999, Baca et al. 1997, Bergamaschi and Putti 1999, Beliaev and Schotting 2001, Rybak et al. 2015, Shah et al. 2022]. In this context, this work studies transient seepage flow in variably saturated porous media. It is a mathematical model developed using the finite difference method to describe the various phenomena that groundwater flow can undergo, such as gravity drainage, infiltration and evaporation. This model is based on Richards' equation.

This study can help to understand transient desaturation phenomena that can occur in a dike after a long period of drought (global warming) and a rapid rise in the water level in the reservoir.

2. Mathematical model

2.1. Assumptions

- the air is at atmospheric pressure (the air phase is connected to the atmosphere),
- the fluid and solid phases of porous medium are incompressible,
- hysteresis effects and thermal processes are neglected.

2.2. Governing equation

By coupling Darcy-Buckinkham law:

$$q = -k(h) \cdot \overrightarrow{grad}H = -k(h) \cdot \overrightarrow{grad}\left(h - \frac{\vec{g}}{g}\vec{z}\right) \tag{1}$$

where:

- q – the Darcy velocity (m/s),
- $k(h)$ – the unsaturated hydraulic permeability (m/s),
- H – the total hydraulic head, defined by: $H = h - z$, h is the capillary pressure, and z is the vertical coordinate that we will now consider positive downwards.

And the law of conservation of mass:

$$\frac{\partial \theta}{\partial t} = -\nabla \vec{q} \tag{2}$$

We obtain the transient flow equation in a variably saturated porous media (Richards equation). This equation can be written in several forms with either pressure head h (m) or moisture content θ (m^3/m^3):

- The fluid pressure based:

$$C(h)\frac{\partial h}{\partial t} = \nabla\left(k(h)\nabla\left(h - \frac{\vec{g}}{g}\vec{z}\right)\right) \tag{3}$$

- The fluid content based:

$$\frac{\partial \theta}{\partial t} = \nabla\left(D(\theta)\nabla\theta - \frac{\vec{g}}{g}k(\theta)\right) \tag{4}$$

- The mixed form:

$$\frac{\partial \theta}{\partial t} = \nabla\left(k(h)\nabla\left(h - \frac{\vec{g}}{g}\vec{z}\right)\right) \tag{5}$$

where:

- $C(h) = \frac{\partial \theta}{\partial h}$ – the moisture capacity (m^{-1}),
- $D(\theta) = \frac{k(h)}{C(h)}$ – the unsaturated diffusivity (m^2/s).

2.3. Hydrodynamic constitutive relationships

It is difficult to solve Richards equation because of the strong non-linearity between the pressure head h and the moisture content θ. However, in order to overcome this linearity, several models have been proposed in the literature to describe the water retention curve θ(h) and the evolution of the unsaturated hydraulic conductivity k= k(h):

- **Modified Mualem–van Genuchten (MMVG) model**:

$$\theta(h) = S_e(h)(\theta_s - \theta_r) + \theta_r \tag{6}$$

$$k(h) = k_s \cdot S_e(h)^{1/2}\left[1 - \left(1 - S_e(h)^{1/2}\right)^m\right]^2 \tag{7}$$

where:

k_s – the saturated hydraulic conductivity,
θ_s and θ_r – respectively, the saturation and the residual moisture content,
S_e – the saturation rate:

$$S_e(h) = \left[\frac{1}{1 + (\alpha \cdot h)^n}\right]^m; \quad m = 1 - \frac{1}{n} \tag{8}$$

where:

α – the inverse capillary length parameter related to $\theta(h)$,
n – parameter related to the distribution of the pore size of the porous medium.

- **Haverkamp (1977) relationships**:

$$\theta(h) = \frac{\alpha(\theta_s - \theta_r)}{\alpha + h^\beta} + \theta_r \tag{9}$$

$$k(h) = k_s \frac{A}{A + h^B} \tag{10}$$

where:

$\alpha = 1.611 \cdot 10^6$,
$\beta = 3.96$,
$A = 1.175 \cdot 10^6$ and $B = 4.74$.

- **Exponential model of Gardner (1958)**:

$$S_e(h) = e^{\alpha(h - h_b)} \tag{11}$$

$$k(h) = k_s \cdot e^{\alpha(h - h_b)} \tag{12}$$

where:

α – a shape parameter (a measure of the pore size distribution of the medium),
h_b – the bubble pressure, which is assumed to be equal 0.

3. Numerical resolution

This h-based form of the Richards equation is solved numerically by the 1D upwind finite difference method. The sample is discretized into a set of elementary layers referenced by the index j and the instant by the exponent i. Explicit time scheme is used to solve this equation:

$$h_j^{i+1} = h_j^i + \frac{\Delta t}{c_j^i \Delta z}\left[k^i_{j+\frac{1}{2}}\left(\frac{h^i_{j+1} - h^i_j}{\Delta z} - 1\right) - k^i_{j-\frac{1}{2}}\left(\frac{h^i_j - h^i_{j-1}}{\Delta z} - 1\right)\right] \quad (13)$$

where: $k^i_{j+\frac{1}{2}} = \frac{k^i_{j+1} + k^i_j}{2}$; $k^i_{j-\frac{1}{2}} = \frac{k^i_j + k^i_{j-1}}{2}$

For this scheme, the space and time increments Δt and Δx must be chosen to verify the CFL condition:

$$\Delta t \leq \frac{\Delta z^2}{\max\left(\frac{k^i_j}{C^i_j}\right)} \quad (14)$$

Therefore,

$$\Delta t \leq \frac{\Delta z^2}{D_{\max}} \quad (15)$$

Of course, the diffusivity D is specifically applicable to partially saturated soils because, in the case of saturated soils (θ = cst =1), the capacity C becomes zero, leading to an infinite value for the diffusivity D.

4. Results and comments

4.1. Sensitivity analysis

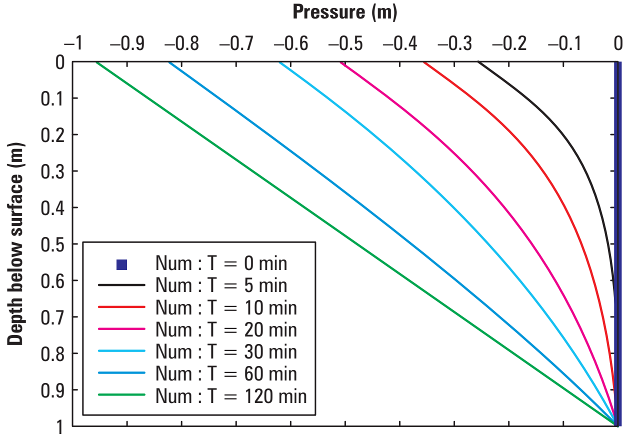
4.1.1. Drainage case

It is a 2m column of sand, initially saturated, above a layer of gravel where water can drain freely (Fig. 3a). Thus, the initial and boundary conditions are as follows:

- Initial condition $t = 0$, $z \geq 0$: $h(z, 0) = 0$ ($\theta(z, 0) = \theta_s$)
- Boundary conditions: $z = 0 : \frac{\partial h(0,t)}{\partial z}$ (no-flow boundary);
- $z = L : h(L, t)$ (constant-head boundary) or $\theta(L, t) = \theta_s$.

Figure 1 shows that the suction decreases spatially from top to bottom (as a function of depth) until it cancels out and increases locally (as a function of time). This is explained by the fact that we are dealing with free drainage, where initially ($t = 0$) the pressure is equal to the atmospheric pressure and locally at the bottom of the sample ($z = L$) the pressure is equal to the atmospheric pressure ($h = 0$) at any time.

Figure 2 shows the evolution of the water content at each level of the soil during the drainage process. It is clear that the water content is equal to θ_s at the initial time (saturated soil) and at the level of the free surface ($z = L$).



Source: Author's own study

Fig. 1. Pressure profiles *h* (*z*) during drainage for different time values of T

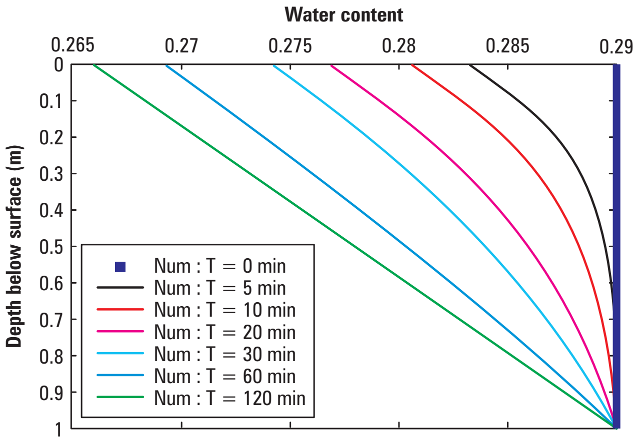


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Fig. 2. Water content during drainage for different time values of *T*

4.1.2. Infiltration case

In this part, we study the case of infiltration applied to a sample 70 cm high and initially subjected to capillary pressure (Fig. 3b). The evolution of this suction is studied according to a chosen time interval of 0.1 hour. The initial and boundary conditions are those provided by Haverkamp et al. [1977]:

- Initial condition t = 0: $h(z, 0) = -61.5$ cm ($\theta = 0.10$)
- Boundary conditions $t > 0$, $z = 0$: $\mathrm{h}(0, t) = -20.73$ cm ($\theta = 0.267$); $t > 0$, $z = L$: $\mathrm{h}(L, t) = -61.5$ cm.

Figure 4 shows the spatial and temporal evolution of the suction between two extreme values, those of the boundary conditions, namely $h = -61.5$ cm and $h = -20.73$ cm. Note that the capillary pressure decreases locally with time, but this evolution begins first from the top of the soil and spreads gradually but slowly downwards with a chosen time interval of 0.1 hour. For example, the soil layer, which is just above the lower boundary layer, only undergoes this variation after approximately 0.8 hours.

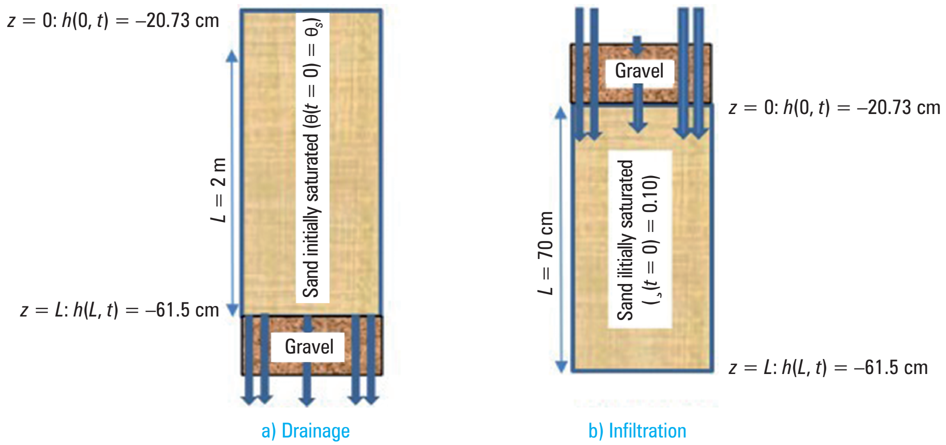

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Fig. 3. Schematic of drainage and infiltration conditions

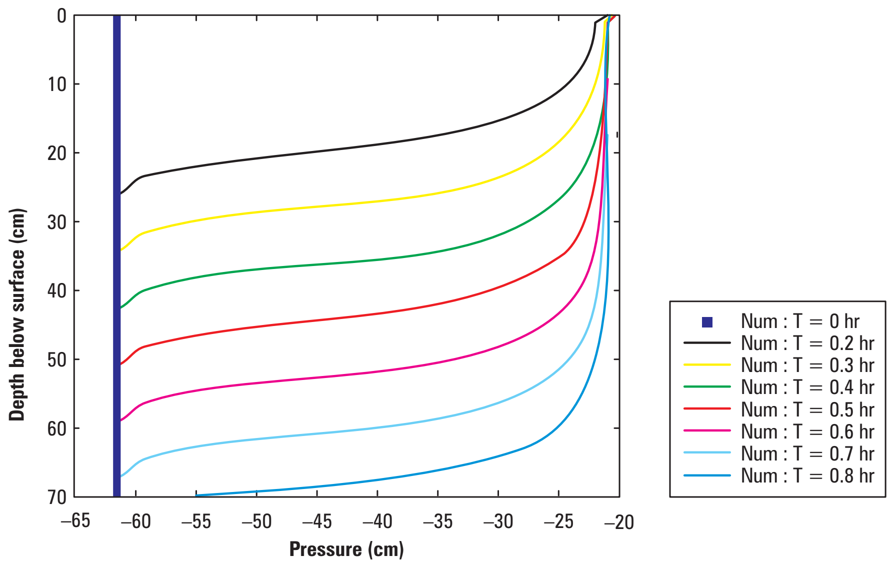

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Fig. 4. Pressure profile during infiltration process

4.1.3. Evaporation case

With regard to evaporation, it involves exposing a soil sample (2 *m* in height) to an evaporation intensity through the surface of the soil. On the other hand, the free surface is maintained at the level of the bottom of the ground. Therefore, the initial and boundary conditions are:

- Initial condition t = 0: $h\ (z, 0) = 0\ (\theta\ (z, 0) = \theta_s)$;
- Boundary conditions: At the surface of soil: $q\ (0, t) = 2.7$ cm/h; and at the bottom: $h\ (L, t) = 0$ (constant-head boundary) or $\theta\ (L, t) = \theta_s$.

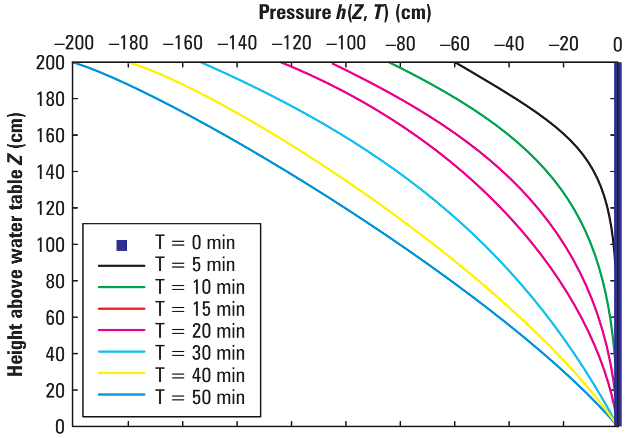

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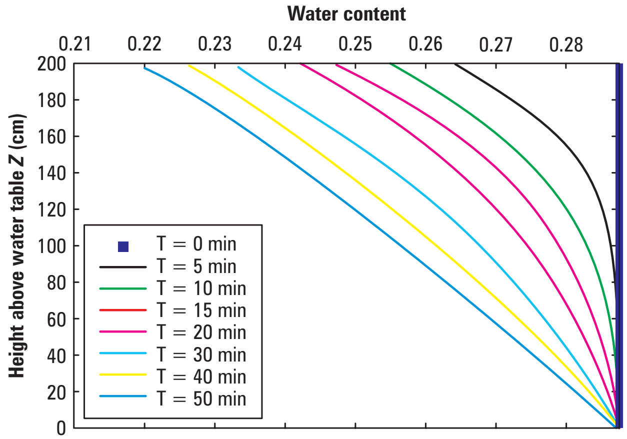
Fig. 5. Pressure profile during evaporation process for different time values of *T*



Source: Author's own study

Fig. 6. Water content profile during evaporation for different time values of *T*

Figures 5 and 6, respectively, show the spatial and temporal evolution of the capillary pressure and the water content profiles during evaporation process. The soil is initially at atmospheric pressure (Fig. 4) (initial state of saturation (Fig. 5)) and as water evaporates capillary pressure begins to spread first through the top of the soil, then it extends downwards (Fig. 4). This is reflected in Figure 5 by a spatial and temporal decrease in water content.

4.2. Experimental validation

4.2.1. Drainage case

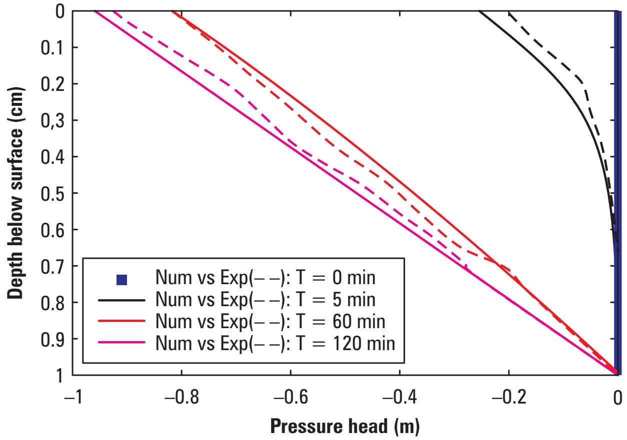
In this study, the drainage experiment carried out by Liakopoulos (1965) was simulated to validate the proposed numerical model. This involves determining the evolution of water pressure during the gravity drainage of a column of saturated Del Monte sand (1 m high). Piezometers were installed in order to measure the temporal evolution of the water pressure at different levels. The sample was allowed to percolate until it reached a state of complete saturation. At the beginning of the experiment, the water supply to the upper end was turned off. Then the pore water of the sand column was gradually drained by gravity from the lower end and air begins to fill the unsaturated pore space of the sand. Liakopoulos measured the temporal evolution of the capillary tension at different levels of the sample.

Figure 7 shows a comparison between experimental data and numerical results of the pressure head obtained after three different times: $T = 5$ mn, $T = 60$ mn, $T = 120$ mn. The experimental data are related to a vertical column drainage experiment realized by Liakopoulos [1965], in which a vertical column with a height of 100 cm was filled with Del Monte sand. Before the experiment, water was continuously poured on top of the column, and allowed to drain freely at the bottom. The inflow was calibrated so that just before $t = 0$ the column was entirely saturated, and zero water pression was established throughout the column, as measured by tensiometers distributed at different levels. At $t = 0$, the inflow was stopped and the water was allowed to flow freely at the bottom. During the drainage process, the pressure distribution was continuously measured by the tensiometers at different times. The question is therefore to study numerically the pressure distribution along the column.

Verified by the mean square error values as shown in Table 1, it becomes apparent that there is a good agreement between the experimental data and the pressure head profiles computed by the proposed code.

Table 1. Mean Square Error (MSE), case: pressure head during drainage process

Time T (mn)	MSE
$T = 5$	0.0010
$T = 60$	0.0019
$T = 120$	0.0004



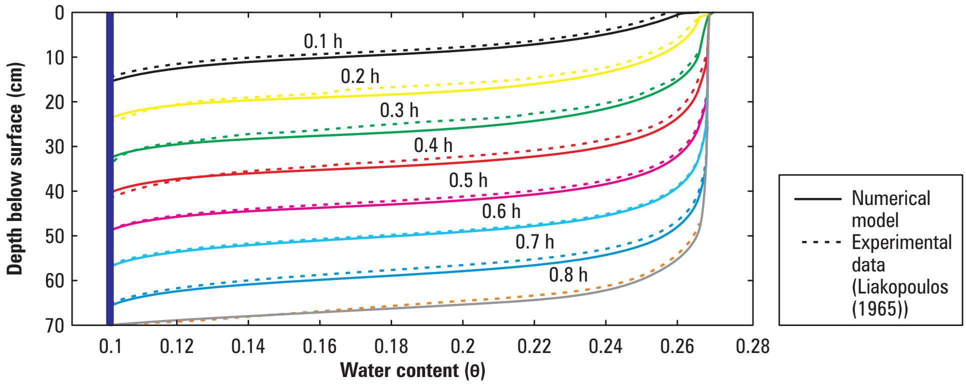
Source: Author's own study

Fig. 7. Pressure head during drainage process: Comparison between the proposed numerical model and Liakopoulos experimental data

4.2.2. Infiltration case

The results of the proposed numerical code were compared with the results of the infiltration experiment carried out by Haverkamp et al. [1977]. The authors subjected a sand column to a constant seepage flow of 3.29 m/day at the top of the column and a constant pressure of h = –0.615 m was maintained at the bottom. This is a sand column with a height of 1 m with the following parameters: k_s = 8.16 m/day, θ_s = 0.287, θ_r = 0.075. The initial and boundary conditions are as follows:

- Initial condition $t = 0, z \geq 0: \theta(z, 0) = 0.10$;
- Boundary conditions: $t > 0, z = 0: q(0, t) = 13.69$ cm/h; $t > 0, z = L: \theta(z, 0) = 0.1$.



Source: Author's own study

Fig. 8. Water content: Comparison between the proposed numerical model and Liakopoulos experimental data

Numerical water content profiles, at time interval of 0.1 hour, are compared with experimental water content data as shown in Figure 8.

5. Conclusion

In this paper, we have developed a numerical model to simulate transient flow in a variably saturated porous medium. This model is based on solving the Richards equation, which is a parabolic and strongly non-linear equation. This non-linear character derives from the non-linear interdependence of the hydraulic permeability with the water content and consequently, with the capillary pressure. The proposed code (numerical model) deals with different flow states such as drainage, infiltration and evaporation. Good agreement with previously published experimental results is evident. Therefore, this numerical model for unsaturated flow calculations provides an accurate representation of the physical reality.

The significance of this model lies in its simplicity and its generality in treating the different flow states in a variably saturated porous medium and therefore its usefulness in practice for engineers in different fields such as hydrogeology, agronomy and geotechnics.

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